HISTORICAL ROOTS OF THE THEORY OF HYDROSTATIC STABILITY OF SHIPS

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Abstract

The physical principles of hydrostatic stability for floating systems were first pronounced by ARCHIMEDES in antiquity, although his demonstration examples were limited to simple geometrical shapes. The assessment of stability properties of a ship of arbitrary shape at the design stage became practically feasible only about two millennia later after the advent of infinitesimal calculus and analysis. The modern theory of hydrostatic stability of ships was founded independently and almost simultaneously by Pierre BOUGUER ("Traité du Navire", 1746) and Leonhard EULER ("Scientia Navalis", 1749). They established initial hydrostatic stability criteria, BOUGUER's well-known metacenter and EULER's restoring moment for small angles of heel, and defined practical procedures for evaluating these criteria. Both dealt also with other aspects of stability theory. This paper will describe and reappraise the concepts and ideas leading to these historical landmarks, compare the approaches and discuss the earliest efforts leading to the practical acceptance of stability analysis in ship design and shipbuilding.

1. INTRODUCTION

Human awareness of the significance of ship stability for the safety of ocean voyages is probably as ancient as seafaring. An intuitive, qualitative understanding of stability and of the risks of insufficient stability must have existed for millennia. The foundations for a scientific physical explanation and for a quantitative assessment of hydrostatic stability were laid by ARCHIMEDES in antiquity ("On Floating Bodies", [1]). Yet despite many important contributions and partially successful attempts by scientists in the early modern era like STEVIN, HUYGENS, and HOSTE among others it took until almost the mid-eighteenth century before a mature scientific theory of ship hydrostatic stability was formulated and published. Pierre BOUGUER ("Traité du Navire", 1746, [2]) and Leonhard EULER ("Scientia Navalis", 1749, [3], [4]) were the founders of modern ship stability theory, who quite independently and almost simultaneously arrived at their landmark results for hydrostatic stability criteria. BOUGUER developed the theory and introduced the terminology of the *metacenter* and the *metacentric curve*. EULER defined the criterion of the *initial restoring moment*, which for hydrostatic stability amounts to an equivalent concept. The full implementation of computational methods for evaluating these criteria and their acceptance by practitioners took even several decades longer.Paradoxically this chain of events raises two nearly contradictory questions, which this paper will address:

o Why did it take so long for these formalized, quantitative criteria for the stability of ships of arbitrary shape to be pronounced?

- When at last the discoveries were made, why then did *two* independent, but equivalent solutions *suddenly* appear almost at the same time?

The knowledge required to evaluate the stability of a ship rests on many concepts and requires many autonomous discoveries to be made and to be brought into concerted application. These include:

- The idea of conceptual experiments in dealing with mechanical systems.
- The abstraction of thinking in terms of resulting forces and moments of weight and buoyancy ("lumped effects") substituted for distributed effects of gravity and pressure.
- The axiom of force equilibrium, here between weight and buoyancy force (Principle of ARCHIMEDES).
- The axiom of moment equilibrium in a system at rest.
- A definition and a test for system stability.
- A method for finding the combined weight and center of gravity (CG) for several weight components.
- A principle for finding the resultant buoyancy force and its line of action (through the center of buoyancy, CB).
- A method for calculating volumes and their centroids, first for simple solids, then for arbitrary ship shapes.
- An analytical formulation of a stability criterion (for infinitesimal and for finite angles of inclination).

Although the physical principles of hydrostatic stability were already established by ARCHIMEDES, it took a long time before the analytical formulation of stability criteria could be pronounced for the general case of ships, essentially by means of calculus, and before numerical evaluations became feasible. BOUGUER and EULER were the first to find an analytical criterion for initial stability, BOUGUER in terms of the *metacenter*. EULER in terms of the *initial restoring moment*. BOUGUER went beyond this in several practical aspects. The intriguing question remains: How did two scientific minds work independently to come to rather equivalent results? What were their sources, their background, their approach, their logic, their justification and verification? Which were their unique original thoughts and how did they differ?

To answer these questions it is not sufficient to look only at their final results and conclusions, but it is necessary to examine more closely the methodical approaches taken to the ship stability issue and to compare the trains of thought by which the individual authors arrived at their results. In this article we will review the developments that led to this historical stage when modern hydrostatic stability theory was founded.

2. PRECURSORS

2.1 Archimedes

ARCHIMEDES of Syracuse (ca. 287-212 B.C.), the eminent mathematician, mechanicist and engineering scientist in antiquity, is also the founder of ship hydrostatics and hydrostatic stability, which he established as scientific subjects on an axiomatic basis. ARCHIMEDES was brought up in the early Hellenistic era in the tradition of Greek philosophy, logical rigor and fundamental geometric thought. ARCHIMEDES was well educated in these subjects in Syracuse and very probably also spent an extended study period in Alexandria, the evolving Hellenistic center of science, at the Mouseion (founded in 286 B.C.). There he met many leading contemporary scientists, e.g., DOSITHEOS, ARISTARCHOS and ERATOSTHENES, with whom he maintained lifelong friendships and scientific correspondence. Steeped in this tradition of Greek geometry and mechanics, ARCHIMEDES learned how to excel in the art of deductive proofs from first premises usually based on conceptual models, i.e., models of thought, by which physical reality was idealized. But ARCHIMEDES was also unique, as many legends on his engineering accomplishments tell us, in applying practical observation to test his scientific hypotheses and to develop engineering appliations, although these achievements are not mentioned in his own written work.

We are fortunate that many, though not all, of ARCHIMEDES' treatises are preserved, all derived from a few handwritten copies made in the Byzantine Empire during the 9th and 10th centuries and transmitted to posterity on circuitous routes to resurface essentially through the ARCHIMEDES revival during the Renaissance. Luckily, one Greek manuscript, which was later lost, had been translated into Latin by the Dominican monk Willem van MOERBEKE. This translation, which was published in 1269 and was later named Codex B, also contained ARCHIMEDES' treatise "On Floating Bodies", Books I and II. It became the only accessible reference to ARCHIMEDES' work on hydrostatics for many centuries until in 1906 most surprisingly an old 10th century palimpsest was rediscovered in a Greek monastery in Constantinople by J. HEIBERG [5]. This palimpsest under the writing of a 12th century monk of a Greek prayerbook had retained significant traces of the rinsed off text of a Greek manuscript from ARCHIMEDES including the Greek version of "On Floating Bodies". ARCHIMEDES' texts were soon reconstructed from this source, transcribed and translated into modern languages (e.g. HEATH [1]). Meanwhile this palimpsest, which had disappeared in private possession in the aftermath of the Greek-Turkish war in 1920-22 has turned up again recently and is under new scientific evaluation at the Walters Art Museum in Baltimore [6]. It is on these sources primarily that we can base a reliable evaluation of ARCHIMEDES' contributions to ship hydrostatics today.

ARCHIMEDES preceded his work on hydrostatics by a number of other treatises establishing certain axioms of mechanics:

<u>The Law of the Lever:</u>
In his treatise "The Equilibrium of Planes" ARCHIMEDES first deals with the equilibrium of moments about a fulcrum in a lever system. Although he claims to deduce this principle from geometric reasoning alone, it is actually understood today that the law of the lever is equivalent to the axiom of moment equilibrium in mechanics. Second, he introduces the concept of "centroids" of quantities (areas, volumes, weights) into which the quantities can be "lumped" as concentrated effects so that moment equilibrium is retained. Third, he proposes a method for finding the "compound centroid" of a system of components, e.g., a center of gravity. Finally, he proves the critical "centroid shift theorem" i.e., a rule for the shift of the system centroid when some quantity is added to, removed from or shifted within the system. All of these concepts and results are essential physical principles as prerequisites for his work on hydrostatics.

<u>Quadrature:</u>
In his treatise "The Quadrature of the Parabola" ARCHIMEDES illustrates by the example of the parabola how the Greeks determined areas and volumes of elementary shapes without the availability of calculus. Here he uses a method, well-known since EUDOXUS [7], based on an inscribed polygonal approximant which is continually refined by interval halving until under the given premises it converges to the given curve within a specified error tolerance. This type of deduction was later called "method of exhaustion". Thus the quadrature problem is reduced to that of evaluating the sum of a finite, truncated or sometimes even infinite series of approximations. This method of quadrature generally is not equivalent to calculus for lack of a limiting process to infinitesimal step width, but has nonetheless

inspired many future developments until this day.

The Method of Mechanical Theorems:
This famed treatise, which was actually also discovered in the palimpsest of 1906, explains how ARCHIMEDES used a reasoning based on principles of mechanics (like moment equilibrium of volume quantities) in deriving geometrical results (like volume centroids). He regarded such findings as propositions for later rigorous deduction by strictly geometrical proofs.

On Floating Bodies:
In this treatise ARCHIMEDES makes use of all these prior results and proceeds to lay the foundations for ship hydrostatics and stability in the following steps [1], [8]:

Book I of this treatise begins with *Postulate 1* describing the properties of a fluid at rest axiomatically.

> "Let it be supposed that the fluid is of such character that, its parts lying evenly (N.B.: at the same level) and being continuous (N.B.: coherent), that part which is thrust the less is driven along by that which is thrust the more and that each of its parts is thrust by the fluid which is above it in a perpendicular direction, unless the fluid is constrained by a vessel or anything else".

Although ARCHIMEDES does not use the word "pressure" and the Greeks did not know that concept in antiquity, he does infer that parts under more pressure would drive parts under less pressure so that a fluid cannot be at rest unless the pressure is uniform at a given depth, while the weight of a vertical column of fluid rests on the parts below it. From these very simple axiomatic premises, which do not permit evaluating the local pressure anywhere in the fluid, ARCHIMEDES is able to derive the principles of hydrostatic equilibrium and stability of floating bodies. This is achieved by considering the equilibrium of the resultant buoyancy and gravity forces and of their moments.

ARCHIMEDES' famous *Principle of Hydrostatics* is stated in Book I, *Proposition 5*:

> "Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced" [1].

The proof of this law, usually pronounced today as $\Delta = \gamma V$, is brilliantly brief and conclusive. In all brevity it rests on the argument that in equilibrium the solid is at rest in a fluid at rest, thus if the body is removed from the fluid and the cavity left by its underwater volume is filled with fluid matter, then the fluid can only remain at rest if the replacing fluid volume weighs as much as the solid, else the fluid would not remain in equilibrium and hence at rest.

In Book II, mainly *Proposition 2*, ARCHIMEDES deals with the stability of hydrostatic equilibrium by treating the special case of a solid of simple shape, viz., a segment of a paraboloid of revolution of homogeneous material whose specific gravity is less than that of the fluid on whose top it floats. In equilibrium it floats in an upright condition. The stability is tested by inclining the solid by a finite angle to the vertical, but so that the base of the segment is not immersed. The equilibrium is defined as stable if the solid in the inclined position has a restoring moment tending to restore it to the upright condition.

For the homogeneous solid this stability criterion is readily evaluated geometrically by examining the lever arm between the buoyancy and the gravity force resultants (Figure 1). The buoyancy force acts through the centroid of the underwater volume B, which ARCHIMEDES finds for the inclined paraboloid from theorems proven earlier. The gravity force or weight acts through the *Center of Gravity* R of the homogeneous solid. Our conventional righting arm, the projection of BR on the horizontal, is positive. Instead of using this stability measure, ARCHIMEDES takes a shortcut for this homogeneous solid by splitting off and removing the weight of the submerged part of the solid Δ_1 and the corresponding equal share of the buoyancy force, which have no moment about B because they both act through B. Thus only the weight of the abovewater section of the solid Δ_2, acting through C, and the equal and opposite buoyancy force increment, acting through B, are taken into account. The centroid C is found from B and R by applying the centroid shift theorem when removing the underwater part from the system. This yields a positive "incremental righting arm" for the force couple of Δ_2, acting through B and C respectively. The restoring moment is thus positive and the solid will return to the upright position.

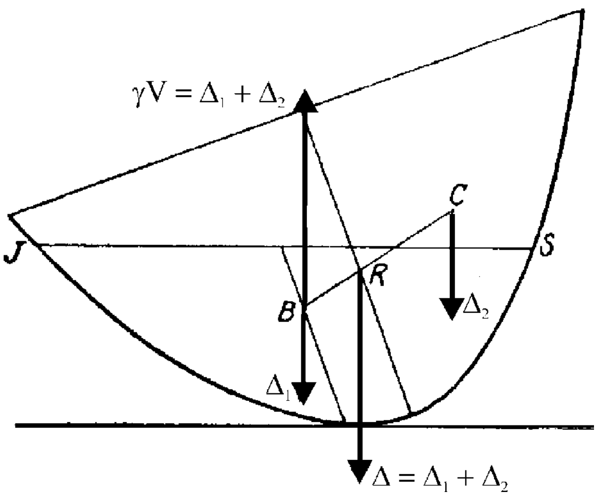

Figure 1: Restoring moments, righting arms for inclined homogeneous paraboloid, based on ARCHIMEDES' "On Floating Bodies", Book II [1] (from [10]).

This application of the hydrostatic stability criterion by ARCHIMEDES is limited to the special case of a homogeneous solid of simple parabolical shape. It demonstrates the physical principles of the hydrostatic stability problem for a finite angle of inclination. It does not extend to floating bodies of arbitrary shape and of non-homogeneous weight distribution, hence actual ships. Still the foundations were laid to enable others much later to treat the generalized case of the ship on the same fundamental grounds.

A more detailed appraisal of ARCHIMEDES' contributions to the hydrostatic stability of floating systems and a thorough historical account is given by NOWACKI [10].

2.2 Stevin and Pascal

In the beginning of the modern era of science the manuscripts of ARCHIMEDES had been rediscovered by several humanists during the Renaissance and were made accessible in print after 1500 by several editions in Greek, Latin and later by translations into modern languages [1], [8], [9].

Simon STEVIN (born 1548 in Bruges, died 1620 in Leyden), the celebrated Flemish/Dutch mechanicist and hydraulic engineer, who knew and admired ARCHIMEDES' works, was probably the first modern scientist who resumed and resurrected the study of hydrostatics and applied it to hydraulics, but also to ships. In his important work "De Beghinselen des Waterwichts" [11], published in Dutch in 1586 and translated into Latin by Willibrord Snellius in 1605, he deals with the principles of hydrostatics and hydraulics. He adopts the idea of "specific gravity" of a fluid from ARCHIMEDES and introduces the concept of "a hydrostatic pressure distribution", as we would call it today, proportional to the weight of a water prism reaching down to the depth in question. This enables him to calculate water loads on walls in a fluid, an important foundation in hydraulic engineering. He also rederives ARCHIMEDES' Principle of Hydrostatics. However when he proceeds to examine the hydrostatic stability of a ship in his later note “Van de Vlietende Topswaerheyt” [11], he correctly reconfirms that for equilibrium buoyancy and weight force resultants must act in the same vertical line through the centers of buoyancy (CB) and gravity (CG). But he erroneously concludes that for stability the CB must always lie above the CG. This error occurred because – unlike ARCHIMEDES - he neglected the centroid shifts resulting from the volume displacement from the emerging to the immersing side in heel. Despite this flaw in an application STEVIN deserves high recognition for founding modern hydrostatics on the concept of hydrostatic pressure.

Blaise PASCAL (born 1623 in Clermont, died 1662 in Paris) is also counted among the modern refounders of hydrostatics. He was familiar with ARCHIMEDES’ and probably with STEVIN’s work. In his “Traité de l’ équilibre des liqueurs”, Paris (1663), he arrives at similar conclusions and justifications regarding the fundamentals of hydrostatics in a fluid as STEVIN did, though he did not deal with ships. The assertion and experimental verification of hydrostatic laws also being applicable to air belong to his original contributions to this subject.

2.3 Huygens

Christiaan HUYGENS (born 1629 in The Hague, died 1695 in The Hague), the eminent Dutch physicist, at the youthful age of 21 made an excursion into hydrostatic stability, which is not well known. In 1650 he wrote a three volume treatise “De iis quae liquido supernatant” [12], in which he applies the methodology of ARCHIMEDES to the stability of floating homogeneous solids of simple shape, reconfirming ARCHIMEDES’ results and extending the applications to floating cones, parallelepipeds, cylinders etc., at the same time studying the stability of these solids through a full circle of rotation. He never published this work in his lifetime because he felt it was incomplete or “of small usefulness if any” or in any case not sufficiently original in comparison to ARCHIMEDES. He wanted the manuscript to be burnt, but it was found in his legacy and at last published in 1908. HUYGENS must be admired for his deep insights into ARCHIMEDES’ fundamentals of hydrostatics and for his own creative extensions. He recognized e.g. that for homogeneous, prismatic solids their specific weight and their aspect ratio are the essential parameters of hydrostatic stability. In his derivations he used a formulation based on virtual work as a principle of equilibrium.

These examples of famous physicists between 1500 and 1700 working in hydrostatics and on hydrostatic stability of floating solids and ships demonstrate that the foundations inherited from ARCHIMEDES were understood by certain specialists, but had not yet been extended or applied to the design and stability evaluation of actual ships. Although the physical principles of stability were understood, the practical evaluation of volumes, volume centroids, weights and centers of gravity for floating bodies of arbitrary shape and non-homogeneous weight distribution de facto still posed substantial difficulties. These were not overcome before the advent and application of calculus during the 18th century.

2.4 Hoste

The French mathematics professor Paul HOSTE, s.j., (1652-1700) was the first to attempt to quantify the problem of ship stability. He did so in his 1697 treatise of naval architecture, *Théorie de la construction des vaisseaux*, which was appended to his book on naval tactics, written at the behest of Admiral TOURVILLE [13]. However, he did not apply calculus to the problem, which was not yet widely known.

HOSTE assumes, without citing STEVIN, that the CG and CB were in a vertical line; but he also allows without proof that the CG could be *above* the CB. However, to explain how this could be possible without the ship tipping over, he also assumes that the buoyancy force was equally divided between the two halves of the ship, forming the base of a triangle. He continues this error by stating:

> "If the center of gravity of the ship is known, the force with which it has to carry sail is easily known, which is no other thing than the product of the weight of the ship by the distance between these centers (of weight and displacement)" [14].

In modern terms, the righting moment is equal to **Δ** x ($\overline{KG}-\overline{KB}$). In other words, the higher the center of gravity, the more stable the ship.

Although he does not provide a theoretical means for determining this "power to carry sail", HOSTE does describe a procedure which could empirically demonstrate this: The inclining experiment. HOSTE asserts that, by measuring the angle of inclination due to suspending a weight from a boom at a certain height, that the "force to carry sail" can be determined [15]. Although HOSTE's argument contains several fundamental errors, it was the first attempt to express the stability of a ship in mathematical terms, and his book remained the only published inquiry into stability for almost half a century.

2.5 La Croix

César Marie de LA CROIX (1690-1747) was the head of administration and finance for the Rochefort dockyard, and maintained the records for the galley fleet. He was not a scientist or engineer. Yet he developed some fundamental concepts on ship motions and hydrostatic stability, derived for a floating parallelepiped and published in 1736 [16], so certainly before BOUGUER's and EULER's work appeared.

LA CROIX was interested in the motions, but also in the hydrostatic restoring moments for this parallelepiped when heeled. He correctly understood the role of weight and buoyancy forces acting in the same vertical line in equilibrium. He also followed ARCHIMEDES in his stability criterion by examining the heeled body and requiring positive righting arms, accounting for the wedge volume shifts. But he falsely determined the influence of the wedge shift moments and hence the righting arms. Besides, since he lacked integration methods based on calculus, he was not able to generalize his results for arbitrary section and waterplane shapes, hence to make them applicable to ships.

Yet although LA CROIX's work was flawed, it may have triggered EULER's renewed interest in ship hydrostatics. In 1735 EULER was asked by the Russian Imperial Academy of Sciences to review the treatise [16], which LA CROIX had submitted there prior to publication. EULER quickly responded and appraised the merits of LA CROIX's problem formulation, but also pointed out the weaknesses of the solution. He then put on record his own correct solution for the initial restoring moment, hence stability criterion, for the parallelepiped (or any other body of constant cross sectional shape), which he claimed to have found earlier [17]. This dates EULER's earliest written mention of the criterion for this special case to 1735. In 1736 in reply to de LA CROIX's rebuttal EULER also provided the complete explicit derivation of this result.

2.6 Sailing vessel propulsion and maneuvering

Questions of ship propulsion, ship motions and maneuvering, but also of ship stability have always aroused an acute interest, not only among seafarers and shipbuilders, but also in the scientific community. The propulsion of sailing vessels by the forces of wind, e.g., was investigated by ARISTOTLE in antiquity and later is associated with such celebrated names of scientists as Francis BACON, Thomas HOBBES and Robert HOOKE, among others.

Toward the end of the 17th century the subject of the propulsion of sailing vessels had gained new scientific interest, also under the influence of the navies in applying scientific principles to ship design and operation. Treatises and monographs on the theory of maneuvering and on the equilibrium of aerodynamic propelling forces and hydromechanic response had appeared, notably by Ignace-Gaston PARDIES (1673), Bernard RENAU d'ELIZAGARAY (1689), Christiaan HUYGENS (1693), Jakob BERNOULLI (1695) and Johann (I) BERNOULLI (1714). This had led to lively controversies, but also to an increasing depth of understanding of the mechanics of the sailing vessel. In fact through the various treatises the principles for applying hydrodynamic under-water and aerodynamic sail forces to the vessel in order to establish its equilibrium position had become well understood. At the same time the importance of allowing for trim and heel (as well as yaw) under these forces and moments had been recognized and dealt with, although the quantitative assessment of the hydrostatic restoring effects remained largely an open issue. Both BOUGUER and EULER would begin their investigations into the nature of hydrostatic stability with the study of propulsion and maneuvering of sailing ships.

2.7 Displacement estimates before calculus

The infinitesimal calculus was important to the development of stability for several reasons, including the practical evaluation of under-water volume and volume centroids. Yet a number of methods to calculate underwater volumes (and therefore displacements) were developed in the 1500s and 1600s, well before any theoretical framework for stability was available to make use of it. Why would shipbuilders go through the trouble of making such calculations? It appears that there were two separate reasons for this: The development of the gunport, and the measurement of cargo tonnage.

The gunport was introduced in the early 1500s, and came into wide use by the middle of the century. This greatly increased the firepower of naval ships, but brought two problems; ships got much heavier, and with large holes in the side of the ship, the available freeboard got dramatically smaller. Thus, the margin for error in estimating the waterline was considerably reduced. To handle this problem, shipbuilders like Anthony DEANE (1638-1721) developed methods to calculate how much armament, ballast and stores should be loaded on a ship to bring it to the correct waterline below the gunports. DEANE and others made use of ship's plans, which were just beginning to be employed by shipbuilders as a construction guide. In his manuscript "Doctrine of Naval Architecture" [18], never published but widely circulated, DEANE demonstrates two methods to calculate the area underneath waterlines at each “bend” or frame of the hull; using either (1) an approximation for the area of a quarter-circle or (2) by dividing the area into rectangles and triangles. DEANE then sums the areas for each frame, multiplies by the frame spacing and multiplies the volume by the specific weight of water to obtain the displacement. He does this for several different waterlines, including the desired waterline below the gunports. When a ship is launched, he can immediately determine the light displacement, and then calculate how much weight should be added to arrive at the design waterline.

A second reason for introducing displacement calculations was to more accurately measure the cargo capacity of a ship. For example, from

1646 to 1669 the Dutch and Danish governments carried on a series of negotiations on cargo measurement. In 1652 a Dutch mathematician Johannes HUDDE identified the basic issue of measuring cargo deadweight by determining the difference between the weight of the ship empty and fully-laden, using a difference-in-drafts method. He suggested to the Dutch authorities that they measure the waterplane areas at each draft of an actual ship in the water (not from plans), by taking measurements to the hull from a line extended at the side of the ship parallel to the centerline. The space between hull and an overall rectangle formed by the length and beam was then divided into trapezoids and triangles, the areas calculated and summed, multiplied by the difference in drafts and multiplied by seawater density, to obtain cargo tonnage. Although the suggestion was never used, HUDDE's cousin Nicolas WITSEN reported it in his 1671 shipbuilding manuscript "Aeloude en hedendaegsche scheepsbouw en bestier" [19].

3. DEVELOPMENT OF STABILITY THEORY

3.1 Bouguer

<u>Biographical Sketch</u>
Pierre BOUGUER (Figure 2) was born on 10 February 1698 in the French coastal town of Le Croisic, near Saint Nazaire at the mouth of the Loire. Educated in the Jesuit school in Vannes, he quickly showed himself a child prodigy. When his father Jean BOUGUER, a royal hydrographer and mathematician, died when Pierre was only 15, he applied for his father's position. After initial hesitation by the authorities, he passed the rigorous exam and was given the post of Royal Professor of Hydrography.

BOUGUER won several Royal Academy of Sciences Prizes for masting, navigation and astronomy before he was 30. He moved to the port of Le Havre in 1731, about the same time he became a member of the Academy. His work caught the attention of the French Minister of the Navy MAUREPAS (1701-1781), who, like COLBERT before him, was convinced of the strategic benefit of ship theory as a way of compensating by quality against the quantitative superiority of the British navy. MAUREPAS supported BOUGUER's research and sponsored his publications.

Figure 2: Portrait of Pierre BOUGUER.

In 1734, BOUGUER became involved in a controversy over the Earth's shape; those who believed in DESCARTES' vortex theories of physics held that the Earth was elongated at the poles, while those who accepted NEWTON's theories of gravitational attraction argued that the Earth was wider at the equator due to centrifugal force. MAUREPAS also held the view that a full understanding of the Earth's shape was essential to navigation, so he sent BOUGUER, along with several other members of the Academy of Sciences (accompanied by two Spanish naval officers) on a Geodesic Mission to Peru to measure the arc length of a meridian at the equator. One of his companions on the Mission was the young Spanish lieutenant Jorge JUAN Y SANTACILIA, who would later become a prominent naval constructor and author of a highly recognized treatise on naval architecture.

BOUGUER spent ten years away (1735-1744), during which time he not only surveyed and calculated a meridian arc length of three degrees of latitude, he also performed various experiments on gravity and astronomy. It was during this Mission, in the peaks and valleys of the Andes far from the ocean, that he wrote much of his monumental work "Traité du navire", the first comprehensive synthesis of naval architecture.

"Traité du navire" was published in 1746, shortly after BOUGUER's return. He remained unmarried, lived in Paris and devoted himself to revising his geodesic work, and carrying out further studies of naval architecture, astronomy, optics, photometry and navigation. BOUGUER died in Paris on 15 August 1758, aged 60. Further details on BOUGUER's biography and scientific work are presented by FERREIRO [20] and DHOMBRES [21].

Early work on ship theory

In 1721 BOUGUER was asked by the Academy of Sciences in Paris to compare the accuracy of two methods of calculating cargo tonnage being proposed to the Council of the Navy for port fees. Although the mathematician Pierre VARIGNON proposed estimating the volume of a ship as a semi-ellipsoid, BOUGUER found that the best results came from a proposal of Jean-Hyacinthe HOCQUART of Toulon, who used a difference-of-waterplanes method that employed equal-width trapezoids to estimate waterplane areas. This was similar to HUDDE's approach but allowed a direct calculation of areas from ship's plans. BOUGUER would later adopt and refine this "method of trapezoids" for his own stability work [22].

In 1727 the Academy offered a Prize for the best treatise on masting, which BOUGUER's entry won. He postulated that a "point vélique", the intersection of the sail force and the resistance of water against the bow, should be directly above the center of gravity to minimize trimming by the bow. In his treatise he relied on HOSTE's theories to explain stability, although where HOSTE implied that the advantage of doubling is through an increase in the center of gravity, BOUGUER invoked ARCHIMEDES to point out that the buoyancy of the added portion of the ship moves the center of buoyancy laterally when heeling, thus increasing the righting moment. Still, this treatise did not yet show any insights into the evaluation of trim or heel angles or restoring moments that he would develop five years later [23].

The metacenter

BOUGUER probably began formulating his theory of stability around 1732, after he had moved to Le Havre, for he tested it using the little 18-gun frigate *Gazelle*, laid down in that dockyard in May 1732 and delivered in January 1734 [24]. However, no letters or manuscripts from that period survive to confirm this, and his derivation of the metacenter would not appear in print until 14 years later in "Traité du navire". The following steps illustrate BOUGUER's derivation of the metacenter in his "Traité du navire" [25].

STEP 1: Premises and axioms

BOUGUER implicitly defines the hydrostatic properties of fluids, based on the principle of hydrostatics that weight and buoyancy are equal, opposite and act in the same vertical line. He does so without proof, without the use of equations and without mentioning ARCHIMEDES by name, although he shows general familiarity with his work. He also implies through geometrical arguments that the pressure of the fluid follows a hydrostatic distribution with depth and is everywhere normal to the surface of the submerged body.

STEP 2: Magnitude of buoyancy force
BOUGUER resolves the submerged surface into very small elements (though without using any calculus notations at this point) and equates the vertical components of the hydro-static pressure forces to the weight of the water column resting on top of the element in the interior of the submerged volume conceptually. Hence, the total pressure resultant is equal to the total weight of water filling the submerged volume. He thus reconfirms the law of ARCHIMEDES by pressure integration.

STEP 3: Measurement of volume and centroids
BOUGUER first suggests two methods for calculating the volume of the ship as a regular solid. The first is to model the ship as an ellipsoid, as originally proposed by VARIGNON, and the second is to divide it into prisms. The set of quadrilateral prisms then forms a polyhedral approximant of the hull surface for volume summation. BOUGUER sees clearly that, by analogy, the same principle of polygonal approximation also holds for evaluating the area of planar curves. He thus quickly homes in on a quadrature rule for curves based on equally wide trapezoids, his favorite rule, which later became known as trapezoidal rule. He uses it to measure areas of waterline "slices" and then combining those two-dimensional slices to obtain a three-dimensional volume. Although he borrowed the idea from HOCQUART's 1717 proposal, BOUGUER refined the method, first by dividing each waterplane into many sections (HOCQUART only proposed three sections), and second by taking the areas of several waterlines to develop the entire volume of the hull (HOCQUART took only one "slice").

BOUGUER then arrives at the conclusion that his quadrature rule is suitable for evaluating the integral of any continuous function of one variable or, if applied recursively, for continuous functions of any number of independent variables. Interestingly he thus interprets the analytical formulation of stability criteria, which he introduces later, as being equivalent to the discretized quadrature rules presented here earlier. The ancestry of his concepts from both practical shipbuilding traditions and modern calculus is still leaving some visible traces.

BOUGUER then digresses for many pages into using the trapezoidal rule to calculate incremental waterlines for estimating a ship's payload capacity, and gives various rules for tunnage admeasurement.

For finding the centroid of the underwater volume, the center of buoyancy CB, BOUGUER then begins by explaining, in very simplistic terms, how to use the sum-of-moments method to determine the centroid of an object. He then derives the area centroid of a planar figure (2D case), then the volume centroid of a solid (3D case), which for a ship he calls the "center of gravity of the hull". He then discusses by example how to evaluate these expressions numerically by the method of trapezoids.

STEP 4: Stability criterion
The center of buoyancy having been determined, BOUGUER next explains why he chooses the metacenter as the initial stability criterion, using the geometrical argument shown in Figure 3.

The ship's center of gravity g is always in the same vertical line as the center of buoyancy Γ, but this geometry is not constant due to the ship's movement. If the ship has a very high center of gravity I, and moves even a little from the upright position (waterline A-B) to another position (waterline a-b), it is no longer statically stable; the center of buoyancy moves from Γ to γ, i.e., away from the vertical of the

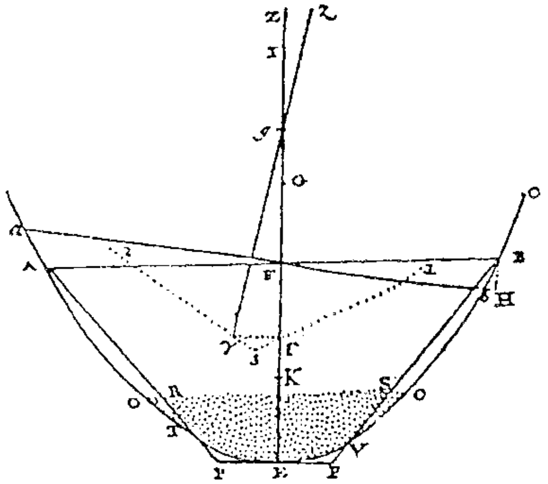

Figure 3: BOUGUER's diagram of the metacenter, [2], fig. 54.

He denotes:

- p = submerged volume
- Γ = upright CB
- γ = inclined CB
- AB = upright waterline
- *ab* = inclined waterline
- I = CG of an unstable ship
- G = CG of a stable ship
- *g* = metacenter
- 1 = centroid of immersed wedge
- 2 = centroid of emerged wedge
- 3 = centroid of body without wedges

center of gravity, and the vertical force of buoyancy shifts from Γ-Z to γ-*z*. The ship's weight, centered at I and on the opposite side of the inclination from the new center of buoyancy, tends to push the ship even further over, rendering it unstable. However, if the center of gravity of the ship is at G, below the intersection *g* of the upright and inclined vertical forces of buoyancy, then the center of gravity is on the *same* side as the new center of buoyancy and the resulting force always tends to restore the ship to the horizontal.

BOUGUER states in his definition of the metacenter [26]:

> "Thus one sees how important it is to know the point of intersection *g*, which at the same time it serves to give a limit to the height which one can give the center of gravity G, [also] determines the case where the ship maintains its horizontal situation from that where it overturns even in the harbor without being able to sustain itself a single instant. The point *g*, which one can justly title the *metacenter* [BOUGUER's italics] is the term that the height of the center of gravity cannot pass, nor even attain; for if the center of gravity G is at *g*, the ship will not assume a horizontal position rather than the inclined one; the two positions are then equally indifferent to it: and it will consequently be incapable of righting itself, whenever some outside cause makes it heel over."

BOUGUER does not use the terms "stable" or "unstable", but rather states that G is either lower or higher than *g*.

Step 5: Evaluation of the criterion

The determination of this point of intersection, BOUGUER says, reduces to the question of the distance between the centers of buoyancy Γ and γ of the submerged body upright and just slightly inclined. BOUGUER employed the all-important shifting of equal immersed and emerged wedges to determine this distance, by drawing a triangle between points 1, 2 and 3. Since the centers of buoyancy Γ and γ lie on the legs of that triangle, the distance Γ – γ must be proportional to the distance between points 1 and 2, and the distance between Γ and point 3 must be proportional to the ratio of the volume of the underwater hull and the wedges. This geometrical explanation sets the stage for the second half of the explanation, the mathematical analysis of the mechanics of stability.

Step 6: Results

In the next section, BOUGUER uses calculus to determine the three unknowns: The distance from point 1 to point 2; the volume of the wedges; and the volume of the hull. BOUGUER imagines that Figure 3 represents the largest section of the ship, which actually extends through the plane of the page in the longitudinal direction x, with the immersed and emerged wedges actually an infinite sequence of triangles of width y (the largest being b = F-B, or the half-breadth of the hull at the waterline) and height e (= H-B) going through the length of the hull at a distance dx from each other; integrating, the volume of the wedge is

$$V_{wedge} = \frac{e}{2b} \int y^2\, dx \qquad (1)$$

The second unknown is the distance between the centers of the wedges, points 1 to 2; but since the center of a triangle is two-thirds the height from apex to base, it is straightforward to obtain this distance as

$$\text{Distance 1-2} = \frac{4 \int y^3\, dx}{3 \int y^2\, dx} \qquad (2)$$

The third unknown, the volume of the hull p, is derived using the trapezoidal rule. Putting the three together, the distance is:

$$\Gamma\gamma = \frac{2\, e \int y^3\, dx}{3\, bp} \qquad (3)$$

Finally, observing that the triangle $\Gamma g\gamma$ (i.e., between the centers of buoyancy and the intersection of their vertical lines of force) is similar to the triangles formed by the immersed and emerged wedges, the height of the metacenter g above the center of buoyancy Γ is found via Euclid to be:

$$\Gamma g = \frac{2 \int y^3\, dx}{3\, p} \qquad (4)$$

This is the now-famous equation for the height of the metacenter.

Implications of the metacenter

BOUGUER develops in thorough detail both the theoretical and practical aspects of the metacenter. He derives the metacenter for various solids (ellipsoid, parallelepiped, prismatic body) and presents the procedures for its practical, numerical calculation for ships. These explanations were detailed enough for practical applications and became the foundation for later textbooks.

But BOUGER also charged ahead beyond the initial metacenter for infinitesimal angles of heel when he introduced the concept of the "métacentrique", i.e., the metacentric curve for finite angles of heel. First, he clearly recognized that the same physical principles and stabilty criteria apply to an inclined position of the the ship as they do for the upright case. Second, he recognized that the metacentric curve for finite angles is in fact the locus of the centers of curvature of the curve of the centers of buoyancy. This was a brilliant, original insight. The locus of the centers of curvature of a curve was known since Christiaan HUYGENS's work on the pendulum clock [27] under the name of "développée" (or evolute). Third, BOUGUER also knew how to construct the evolute of a given curve. For a wall-sided ship (or parallelepiped) the metacentric curve is a cusp shaped curve composed of two hyperbolas lying above the metacenter, as BOUGUER showed. His demonstration examples for the "métacentrique" do document that he understood how to approach stability for finite angles of heel, though he never used a "righting arm" criterion.

On practical aspects of initial stability he recommends that the widest point of the ship be no lower than where the maximum heel would be before it begins to tumble-home, or even that ship sides be straight or flared throughout. In other words, BOUGUER was advising to avoid tumble-home altogether, although in somewhat oblique terms.

BOUGUER next provides a numerically-worked example of the application of the metacenter to a real ship, underlining the importance of Gg, the distance between the center of gravity and the metacenter. Using the *Gazelle* as his model, BOUGUER explains how to account for the weight of each part of the ship, including the frames, planking, nails, etc., how to measure the center of gravity of each piece using the keel as the reference point, and how to sum the moments to obtain the overall center of gravity G. He then calculates displacement and center of buoyancy of the ship using the trapezoidal rule, which enables him to find the metacenter *g*. His calculations confirmed that, once properly ballasted, the *Gazelle* would be stable.

BOUGUER continues for almost 50 pages to outline the practical implications of the metacenter on hull design and outfitting. In many cases the implications are re-statements of what constructors already knew; but BOUGUER provided for the first time a rigorous analysis of why they were true. Some of the main points he brings out are:

- The greatest advantage to stability lies in increasing the beam. BOUGUER claims that the "stability" varies as the cube of the beam, by which he must have been referring to the transverse moment of inertia I_T, while the metacentric radius requires closer scrutiny. But he wanted to emphasize the rapid increase of stability with beam. This also explains for the first time why doubling a ship improves stability, although BOUGUER does not explicitly state so.
- Stability is improved by diminishing the weight of the topsides; though this was well-understood, BOUGUER details how to accurately assess the effects.

BOUGUER also correctly describes for the first time how to evaluate the inclining experiment, for whose basic idea he credits HOSTE, in order to determine G*g*, the distance between the center of gravity and the metacenter. In Figure 4, he demonstrates how to use a known weight suspended from a mast to incline the ship and measure the angle of heel. Using the law of similar triangles, he shows that the ratio of the ship's displacement and the suspended weight is proportional to the ratio of G*g* to the angle of heel and distance the weight moves; on the assumption of small angles of heel this allows G*g* (i.e., the reserve of stability) to be calculated directly without knowing the exact position of the metacenter.

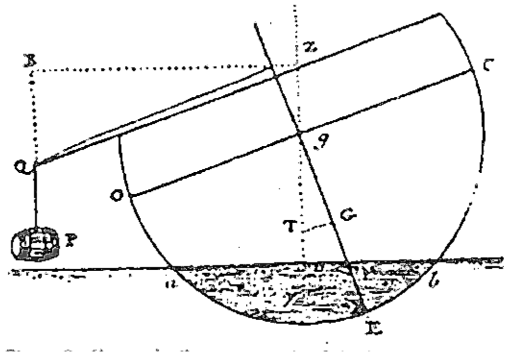

Figure 4: BOUGUER's diagram of an inclining experiment, [2] fig. 55.

BOUGUER demonstrates how to use his metacenter by detailing the calculations of weights and centers of gravity for *Gazelle* while it was still being framed out. It is therefore curious that he did not verify this by performing an inclining experiment on *Gazelle*. It would appear that, while BOUGUER was completing his initial stability work in 1734, he was caught up in the events that led to the Geodesic Mission, and would not return to the subject until he was in the mountains of Peru.

3.2 Euler

Biographical sketch

Leonhard EULER (Figure 5) was born in Basel, Switzerland on 15 April 1707. He was the son of Paulus EULER, parson of the Reformed Church, and his wife Margaretha née BRUCKER. He went to a Latin grammar school in Basel and, as his father recognized his talent early, took private lessons in mathematics. In 1720 he enrolled at the University of Basel as a student and later as a young scientist, the first three years in philosophy where he received a Magister's degree, then in theology. But he soon turned his main interest to mathematics and mechanics, which he studied under Johann (I) BERNOULLI, who was recognized as a leading mathematician of that era. In fact Johann (I) BERNOULLI, who was 40 years EULER's senior, saw EULER's maturing genius and invited him to join the Saturday afternoon "privatissimum" in mathematics, a private circle held in the BERNOULLI's home, where he also met and made friends with younger members of the BERNOULLI family, notably Niklaus (II), Daniel (I) and Johann (II) BERNOULLI. In this period Johann (I) BERNOULLI laid the foundations from which EULER's later mathematical fame developed.

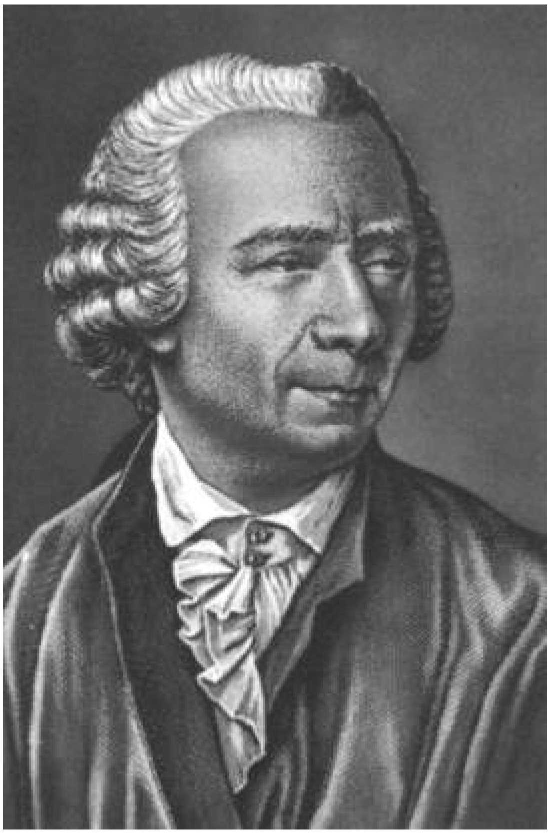

Figure 5: Portrait of Leonhard EULER

In 1727 EULER, who had been looking for an academic position, received an offer from the Russian Imperial Academy of Sciences in St. Petersburg, which Tsar Peter had initiated and which had been opened in 1725, where Niklaus (II) and Daniel (I) BERNOULLI held appointments as professors from this beginning. EULER accepted this prestigious offer, starting as an Adjunct (élève) for a modest salary, and arrived there – after a three month voyage by coach and by ship- in June 1727. EULER began his scientific career in St. Petersburg very successfully, advanced to a professorship in physics and full membership in the Academy in 1731, and spent his "First St. Petersburg Period" (1727-1741) very productively, working on a wide range of scientific subjects and publishing more than 50 treatises and books.

In 1741, amidst political transitions and uncertainties in Russia, EULER received an attractive offer from King FREDERICK II of Prussia to come to Berlin and to work for the Royal Academy of Sciences, which was to be founded. EULER moved there in 1741, continued his illustrious scientific work on an increasing variety of subjects, and when the Academy at last opened in 1746 soon became recognized as one of its most eminent members and as the leader of the mathematical class. He remained immensely productive and published some of his most famous books and treatises during his period in Berlin from 1741 to 1776.

Despite EULER's scientific fame and the considerable merits he earned in the Berlin Academy during its formative first two decades he never developed a relationship with King FREDERICK that made him feel recognized, appreciated and understood. This was one contributing factor why after 25 years in Prussian service he decided to return to St. Petersburg. There he was welcomed and honored at all levels, and spent his "Second St. Petersburg Period (1766-1783)" in relative comfort, peace and recognition despite his weakening health and fading eyesight. His scientific creativity never ceased. A large share of his more than 800 treatises and books also stems from this second period in Russia. His work encompasses the full gamut of scientific topics in his era, not only in mathematics and mechanics, but also in other branches of physics, astronomy, ballistics, music theory, philosophy and theology. He died in St. Petersburg in 1783.

More detailed accounts of EULER's life and scientific vita are given by BURCKHARDT et al. [28], FELLMANN [29] and CALINGER [30].

Early work in ship theory

In 1726 the Académie des Sciences in Paris invited contributions to a prize competition on the optimum placement, number and height of masts in a sail propelled vessel. EULER as a student and protégé of Johann (I) BERNOULLI was of course familiar with the earlier treatises and disputes that had arisen about the forces acting on a sail propelled maneuvering vessel between RENAU, HUYGENS, Johann (I) BERNOULLI a.o. So he felt well enough prepared, also encouraged by Johann (I) BERNOULLI, to submit a treatise ("De Problemate Nautico…", [31]) in 1727, which gained honorable mention, an "accessit" equivalent to a second prize. Pierre BOUGUER was awarded the first prize. In this treatise EULER, who still treads in the foot-steps of Johann (I) BERNOULLI, does not yet show deeper insights into ship hydrostatics although he does draw attention to the requirement that for a ship before the wind the propelling sail force is limited by the acceptable forward trim angle. But he has no reliable physical basis for estimating that angle. Yet this early experience in his life at the age of 20 gave EULER a first thorough acquaintance with the mechanics of ships, familiarized him with nautical and ship construction termi-nology and created in him a lasting interest in ships as topics to be studied by methods of mathematical analysis and engineering mechanics.

In St. Petersburg EULER soon had ample opportunities to return to these subjects. It is not clear when he first occupied himself more deeply with ship hydrostatics. But evidently the appearance of LA CROIX's treatise on the hydrostatic stability of a parallelepiped in 1735 found EULER well prepared, when the Academy asked for his review, to detect certain flaws in LA CROIX's derivations and to present the correct result for this simple shape, which first established EULER's *initial restoring moment criterion* on hydrostatic stability. In his second review in 1736, in response to LA CROIX's rebuttal, EULER went beyond his first results, dealt with some other prismatic shapes (trapezoidal and triangular prisms) and alluded to his general approach to stability [17]. This evidence is sufficient to date EULER's earliest written mention of the hydrostatic stability criterion.

There is also some indirect evidence of EULER's earliest results on hydrostatic stability in his correspondence with the Danish naval constructor and naval attaché in London, Friderich WEGERSLØFF, which he maintained between 1735 and 1740. In a letter of 14 September 1736 WEGERSLØFF acknowledges receipt of a solution for the hydrostatic stability problem, on 22 May 1738 EULER replies and gives a derivation of his stability results and also mentions some experimental verification [32]. It is not clear from the context what sort of tests he had performed.

By 1737 EULER's interests in ship theory were well known at the Imperial Academy. Thus - probably not without his own prior knowledge or suggestion - he was commissioned by the Academy to write a book on this subject, resulting in his monumental two-volume "Scientia Navalis". This work encompasses a presentation of the complete scope of ship theory according to the state of the art at that time and includes many new findings and derivations by EULER. The chapters in the two parts deal with ship hydrostatics, ship resistance and propulsion, maneuvering and ship motions. Many results have been of lasting value and have served as foundations for the growing scientific body of ship hydromechanics. According to EULER's own report in the Preface of "Scientia Navalis", he had worked on this manuscript from 1737 to 1740 [33]. When he left St. Petersburg in 1740 he had completed the first part, which contains all four chapters on hydrostatic equilibrium and stability, and half of the second part. The remaining sections of Part 2 were finished by 1741 in Berlin. Unfortunately, he had much difficulty finding a publisher for this voluminous opus, which thus was not published until 1749 by the Academy in St. Petersburg.

Comprehensive appraisals of EULER's overall contributions to ship theory and related matters are given by HABICHT [34], MIKHAILOV [35] and TRUESDELL [36].

The initial stability criterion

To understand the history of EULER's involvement in ship theory it is important to read the Preface to his "Scientia Navalis". Here he explains that his work will go beyond the established disciplines of hydrography or nautical science and will concentrate on a physical and analytical investigation of the mechanics of ships, at rest and in motion, for which fundamental theoretical works were still missing at that time. In hydrostatics EULER departs from the Principle of ARCHIMEDES, to whom in the preface of "Scientia Navalis" he gives credit and praise. But he adds that the hydrostatic stability of ships must be newly approached and quantified in order to be able to distinguish between stable and unstable equilibrium of ships at the design stage. The experience of naval architects ("architecti navales") alone, long established as it may be, will not be sufficient to prevent unexpected stability accidents.

EULER acknowledges the motivation he received from reviewing de LA CROIX's treatise in 1735-1736, which prompted him to investigate more profoundly the transverse and longitudinal stability of ships. EULER also gives very favorable credit to BOUGUER's "Traité du Navire", which had appeared 3 years before "Scientia Navalis", but he takes much care also to avert any suspicion of plagiarism by calling on the Imperial Academy as witnesses that he had written "Scientia Navalis" essentially between 1737 and 1741 during which period there was no communication with BOUGUER who was in Peru. These statements were never disputed between the two authors, who corresponded in a respectful and amicable fashion on other subjects later.

EULER's derivation of his stability criterion in "Scientia Navalis" proceeded in the following steps [37]:

STEP 1: Premises and axioms

In the first chapter of Book I, EULER deals with the equilibrium of bodies floating in water at rest. In essence he rederives the Hydrostatic Principle of ARCHIMEDES from the modern viewpoint of infinitesimal calculus by integration of the hydrostatic pressure distribution prevailing in a fluid over the surface of the body. The properties of the fluid at rest and the use of calculus for this purpose both were new at the time when EULER wrote these passages. As for the pressure distribution he states axiomatically in the opening paragraph of his book:

> "Lemma: The pressure which the water exerts on the individual points of a submerged body is normal to the body surface; and the force which any surface element sustains is equal to the weight of a straight cylinder of water whose base is equal to the same surface element and whose height is equal to the depth of the element under the water surface".

These brief axioms, viz. the normality of pressure to a surface and the inferred equality of the pressure at a point in a given depth in all directions, were the first analytical formulation for the properties of the fluid and are regarded as the necessary and sufficient conditions for the foundation of hydrostatics [38].

STEP 2: Magnitude of buoyancy force
From these premises EULER proceeds to define by integral calculus the buoyancy force as the pressure resultant and the center of buoyancy (EULER calls the CB: "centrum magnitudinis"), through which it acts, as the volume centroid of the submerged part. He reconfirms also that for equilibrium the buoyancy and weight forces must act in the same vertical line and must be equal in magnitude and opposite in direction. He then illustrates these principles by examples of simple shapes like parallelepipeds and prisms of triangular and trapezoidal cross section. For each of these solids he finds the possible equilibrium conditions over a full circle of rotation as a function of the specific weight of these homogeneous solids, not unlike HUYGENS in his unpublished treatise of 1650.

In Chapter II, briefly digressing from hydrostatics, EULER discusses the resulting motions of a floating body if it is temporarily displaced from its "upright" equilibrium position. Here he explains the "lumping" of masses in their center of gravity (CG), introduces the definition of the principal axes of inertia for ships, underscores the significance of the CG as a reference point, e.g., for decomposing a resulting motion into the translation of the CG and the rotation about it. EULER will adhere to the CG as his system reference point, also when returning to ship motions later.

STEP 3: Measurement of volumes and volume centroids
In stark contrast to BOUGUER, EULER confines himself to analytical definitions of stability criteria, volumes, centroids, areas, moments of inertia etc. He does not address their numerical evaluation at all. He is taking it for granted that once the shape of the ship or body is defined by some function the integrations can be readily performed. In his examples he usually deals with simple shapes where the integrations can be performed in closed form.

STEP 4: Stability criterion
In Chapter III, the main chapter on hydrostatic stability, EULER defines his stability criterion right away (*Proposition 19)* by:

> "The stability which a body floating in water in an equilibrium position maintains, shall be assessed by the *restoring moment* if the body is inclined from equilibrium by an infinitesimally small angle".

EULER illustrates this principle by discussing cases of unstable, neutral and stable equilibrium of ships and adds that it is necessary to quantify stability in terms of the restoring moment because even a stable ship may be in danger by external heeling moments and may require righting moments of greater magnitude. (The issue is not only whether the ship is initially stable or not, but also how much "stability capacity" it has).

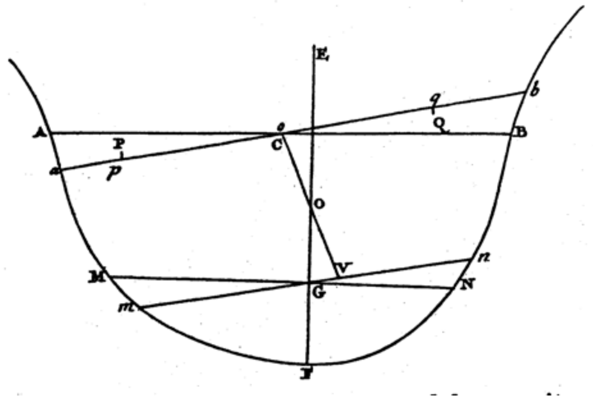

Figure 6: EULER's figure for centroid shift in inclined cross section, 2D case, [3] fig. 39.

STEP 5: Evaluation of the criterion
EULER then enters into the determination of the restoring moments by first examining a planar cross section of arbitrary shape (or thin disk) floating upright (Figure 6). The Figure AMFNB = "AFB" is inclined by a small angle

dw so that the new floating condition has the waterline ab. Since the center of gravity G remains his reference point, also for later purposes, he draws the parallel lines MN and mn to the waterlines before and after inclination, also through G. The center of buoyancy of the cross section is designated by O. A normal VOo to the inclined waterline is drawn through O. The equality of the immersed and emerging wedges requires that ab intersects AB at the center point C so that AC = BC. Let the displacement per unit length and the equal buoyancy force of the cross section before inclination be denoted by M = γ (AFB). For the inclined position the restoring moment is composed of three contributions:

1. The effect of the original submerged volume forming a positive restoring couple of forces through G and V:

$$M\,GV = M\,GO\,dw \quad (5)$$

2. The effect of the submerged wedge CBb whose cross section area is:

$$\frac{BC^2 dw}{2} = \frac{AB^2 dw}{8} \quad (6)$$

and whose restoring moment about G hence is:

$$\gamma \frac{AB^2 dw}{2}(qo + GV) \quad (7)$$

where $qo = \frac{2}{3} Cb = \frac{1}{3} AB$

3. Likewise for the emerging wedge ACa the restoring moment is

$$-\gamma \frac{AB^2 dw}{8}(po - GV) \quad (8)$$

where $po = \frac{2}{3} Ca = \frac{1}{3} AB$

For all moments combined, replacing γ by M/(AFB):

$$M_{REST} = M\,GO\,dw + \frac{M(AB^2 dw)(po + qo)}{8AFB}$$

$$= \quad Mdw\,[GO + \frac{AB^3}{12AFB}] \quad (9)$$

The expression in square brackets has the dimension of a length and is the now well known result, corresponding to, in BOUGUER's later terminology, the *metacentric radius* of the cross section. This is how EULER's *restoring moment* and BOUGUER's *metacentric* radius are connected. Note that the GO term reverses sign if G lies above the center of buoyancy O, as is common in cargo ships. EULER discusses this result at some length and for several simple shapes.

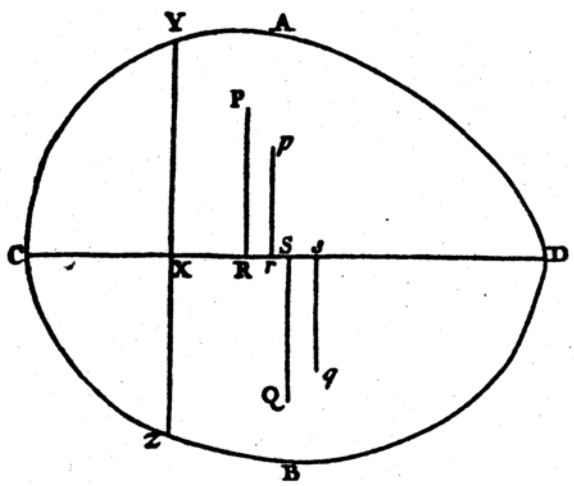

Figure 7: EULER's figure for the derivation of the stability criterion for a body of arbitrary shape, 3D case, [3] fig. 48.

In *Proposition 29* EULER then arrives at the general three-dimensional case of a floating body of arbitrary, in particular asymmetrical shape, whose waterplane is drawn in Figure 7. He denotes:

M = displacement or weight of body
V = submerged volume
GO = distance between CG and CB, positive for G above B
CD = reference axis of waterplane through centroid of waterplane area, parallel to axis through G
CX = x = abscissa from origin C
XY = y = ordinate in upper part of waterplane
XZ = z = ordinate in lower part of waterplane

p and q = area centroids of upper and lower parts of waterplane
P and Q = equivalent pendulum lengths of waterplane area parts w.r.t. axis CD

With

$$pr = \frac{\int y^2 dx}{2\int ydx} \qquad qs = \frac{\int z^2 dx}{2\int zdx}$$

$$PR = \frac{2}{3}\frac{\int y^3 dx}{\int y^2 dx} \qquad QS = \frac{2}{3}\frac{\int z^3 dx}{\int z^2 dx}$$

Euler derives, since the axis CD runs through the waterplane centroid:

$$\int y^2\, dx = \int z^2\, dx, \qquad \text{and hence}$$

$$PR + QS = \frac{2}{3}\frac{\int (y^3 + z^3)dx}{\int y^2 dx} \qquad (10)$$

from which in analogy to the planar case the *restoring moment*, *divided by the angle dw*, for the ship becomes:

$$M\,(GO + \frac{\int (y^3 + z^3)dx}{3V}) \qquad (11)$$

In the special case of port/starboard symmetry (y = z), with

$$I_T = \frac{2}{3}\int y^3\, dx; \text{ and } \frac{I_T}{V} = OM$$

In our familiar notation (GO= $\overline{GB}$, OM= $\overline{BM}$) the restoring moment simplifies into:

$$M\,(\overline{GB} + \overline{BM}) = M\,\overline{GM} \qquad (12)$$

STEP 6: Results
This summarizes the course of steps EULER took to derive the *initial restoring moment.* EULER never used the word *metacenter.* EULER's result thus is the *initial restoring moment*, divided by (M dw), which he uses as his stability criterion.

Again EULER illustrates this result by many examples for simple shapes of solids, even by an analytical formulation for a shiplike body with parabolic section shapes. But he does not present any numerical calculations for an actual ship although he discusses many practical implications of his results.

Implications of the stability criterion

EULER interprets his results for practical applications to ships, mainly in Part II of "Scientia Navalis". For EULER ships are "floating objects, carrying a cargo or payload, symmetrical starboard and port, propelled by rowing and/or sails". Among the major conclusions he draws, we quote:

- Stability is judged by the restoring moment divided by the angle dw.
- Transverse and longitudinal stability are clearly distinguished and both addressed. EULER also derives an expression for combined heel and trim under oblique sail force moments, deriving the oblique restoring effects by rotation of the reference axis from the principal axes.
- To assess stability you need to know the ship's displacement, the centroids CB and CG, the waterplane area and shape, which infers the restoring moment (without using the metacenter).
- To improve stability, lower the CG, raise the CB and/or widen the beam.
- The addition, removal and internal shifts of weight as well as the role of ballast for stability are discussed.
- The use of doubling ("soufflage") with its pros and cons is mentioned.
- In Chapter IV of Part 1 EULER addresses practical problems of inclining ships under external moments or by internal weight shifts, though again only analytically, not numerically.
- He also addresses the issue of external wind loads and required stability, especially for small ships vs. big ships, cargo ships vs. war ships.

EULER finally also makes the interesting "philosophical" remark that sailors knew about measures of stability all along. When the sailors say, "This ship can sustain a strong wind in its sails", then EULER claims they mean the same thing as he expresses by his restoring moments.

EULER's contributions to ship hydrostatics are of lasting value and are based on his strengths in physical intuition and analytical perception.

3.3 Comparison of approaches

Based on the facts presented above it is now possible and in fact intriguing to compare the approaches taken by BOUGUER and EULER in their creative work on hydrostatic stability. This will bring out certain commonalities, but also underscore the differences. We have examined several aspects.

Chronology:
BOUGUER worked on the subject of ship hydrostatic stability from about 1732 to at least 1740, and his treatise was published in 1746. EULER was engaged with this topic between about 1735 and at least 1740, "Scientia Navalis" appeared in 1749. During these formative periods of their new concepts there existed no contacts and no communications between them. Thus their work originated independently.

But after their first scientific acquaintance through their participation in the 1727 Paris Academy award competition it is likely that they both underwent a common fermentation period on related subjects. Probably they were both looking for the missing pieces in the mosaic of sailing ship maneuvering, among which the hydrostatic response was foremost. It appears that their deeper familiarity with ARCHIMEDES originated during the period between 1727 and 1735.

Sources
BOUGUER writes that he was familiar with the work of PARDIES, HOSTE, RENAU, most likely also with relevant publications by HUYGENS and the BERNOULLIs. He shows close acquaintance with ARCHIMEDES' principles, although he does not mention his name. On calculus he quotes the well-known treatise by NEWTON and Roger COTES, but very likely he also thoroughly read de l'HÔPITAL, whose notation in the LEIBNIZ style he preferably uses. His knowledge of calculus was mostly self-taught, based on occasional tutorials and recommendations from mathematics professor Charles René REYNEAU, who wrote the basic textbook on the subject, "Analyse démontrée" in 1708.

EULER also knew ARCHIMEDES, HOSTE, RENAU, whom he mentions. He learnt calculus first-hand from the BERNOULLIs, who stood firmly in the LEIBNIZ tradition. EULER's book on mechanics, "Mechanica" in 2 vols., where he displays exquisite knowledge of the principles of equilibrium and motions of mechanical systems, had appeared in 1736.

Although both authors are not very generous in giving references on their sources it is possible to a certain extent to trace their intellectual ancestry by looking at their scientific and technical terminologies, even if they wrote in different languages, French and Latin. We have analyzed their vocabularies in their main treatises on hydrostatic stability and identified commonalities and distinct differences. They shared much common ground by using such established words as *equilibrium, stability, weight, buoyancy, specific weight, centers of gravity and buoyancy, inclination* (derived from ARCHIMEDES) and *pressure, hydrostatics* (from STEVIN and PASCAL). BOUGUER is unique in inventing the *metacenter* and using the *evolute* (HUYGENS), but never the word "*restoring moment*", while EULER's vocabulary was vice versa. This indicates where their approaches differed.

Stability criteria

Sections 3.1 and 3.2 have shown that BOUGUER and EULER derived equivalent stability criteria, but used different approaches. To sum up the comparison:

Premises and axioms: BOUGUER's work is founded on the premises of ARCHIMEDES, though augmented by a hydrostatic pressure law (STEVIN). EULER defines the axioms of hydrostatics a bit more strictly by postulating the direction independence of pressure and its normality to a surface.

Magnitude of buoyancy force: Both agree and reconfirm the law of ARCHIMEDES by pressure integration and calculus.

Measurement of volumes and centroids: BOUGUER takes the route of establishing numerical procedures, based on quadrature rules, for measuring volumes, centroids, areas etc. before giving analytical definitions of the stability criterion. In passing he skillfully adapts the "method of trapezoids" to shipbuilding applications. EULER skips this step entirely and drives directly at analytical formulations for his stability criterion.

Stability criterion: BOUGUER's invention and choice for a measure of stability is "*metacentric radius",* a geometric quantity representative of initial stability capacity. EULER chooses the *"initial restoring moment"*, a physical quantity, also measuring the ship's initial stability capacity. Although the two ideas are closely connected, their meanings are conceptually different. BOUGUER never refers to "*restoring moments*", EULER never uses "*metacenter*". Neither author recurs to ARCHIMEDES' idea of a "*righting arm*".

Evaluation of the criterion: BOUGUER derives the metacenter mainly from geometric arguments and, allowing for the shift of wedge volumes and hence CB, determines the metacenter as the point of intersection of the upright and inclined buoyancy directions. EULER uses physical arguments of moment equilibrium and shift of volumes and finds the restoring moment simply by a summation of moments.

Results: BOUGUER's result is metacentric radius, a distance. EULER's result is a restoring moment. The two results are equivalent in practice, but not equal in approach.

Clientele, style and language:

BOUGUER's "Traité du navire", the first modern synthesis of theoretical naval architecture, is written in French for a readership of scientists and constructors who are to be introduced to the use of theory in practical ship design and shipbuilding. BOUGUER benefited from both his experience as a hydrographer and his collaboration with ship constructors. His style is clear and logical, explaining many practical details, almost as in a textbook. He is concerned about how this methodology can be implemented. This style does not cause any sacrifices in rigor. His language is lucid, his own new ideas come across crisply.

EULER in his Latin text of "Scientia Navalis" writes as an applied mechanicist/physicist and addresses mainly the scientific community of his era. Among those who were educated to read scientific treatises of this sort in Latin, which was still a lingua franca in the academic world, EULER's brilliance of style, his inimitable logic and clarity were highly praised. These attributes also apply to his "Scientia Navalis". His work on hydrostatic stability is still valid today. But at the same time he did not have any experience with practical shipbuilding applications. Thus he left many details for implementation by later successors, which delayed the spreading of his methods, certainly in comparison to BOUGUER.

Yet in the final balance the scientific and engineering community must be grateful to both BOUGUER and EULER.

3.4 Synopsis

Table 1 gives a synopsis of important milestones and ideas in the development of the theory of hydrostatic stability by indicating where certain elements of this theoretical knowledge first occurred. The importance of ARCHIMEDES's physical insights and the relevance of BOUGUER's and EULER's parallel discoveries described by calculus become clearly visible in this condensed tabular summary of historical steps.

4. FURTHER WORK

4.1 Practical applications

Stability theory quickly found direct applications in the day-to-day practice of ship design. This occurred in two ways: first, by the increasing sophistication of calculations for weights, centers of gravity and the metacenter within the design process; and second, by the use of inclining experiments to validate stability. These applications were surprisingly widespread; stability theory was quickly incorporated in navies where there was already a strong institutional development of scientific naval architecture, notably in France, Denmark, Sweden and Spain. Those countries also created schools of naval architecture during the 18th century, where students were weaned on stability theory and naturally used it when they became constructors at the dockyards. This was not the case for the British or Dutch navies, which had provided little direct support for scientists working on ship theory, and which did not establish permanent schools of naval architecture until the 19th century.

Stability calculations during design

The most famous of these schools of naval architecture was the *Ecole du Génie Maritime*, established in 1741 in France by the scientist Henri Louis DUHAMEL DU MONCEAU (1700-1782), whom MAUREPAS had nominated as Inspector-General of the Navy.

DUHAMEL worked with BOUGUER to create the first comprehensive textbook for the students, based on "Traité du navire". DUHAMEL's genius was to take BOUGUER's complex mathematics and render them into step-by-step instructions on how to calculate the metacenter. The textbook, "Elémens de l'architecture navale" [39] became the standard reference for both students and constructors.

DUHAMEL DU MONCEAU was also involved in developing the Ordinance of 1765 under Navy minister Etienne François, duke of CHOISEUL, which created the *Corps du Génie Maritime* and formalized the data to be included on ship's plans: "centers of gravity and resistance, and height of the metacenter" as well as accompanying calculations and a tabulation of hull materials. Later, the standardized plans introduced in 1786 by DUHAMEL's successor as Inspector-General, Jean-Charles de BORDA, listed the specific immersion of the hull at full and light load in tonnes-per-cm equivalent, which allowed a rapid estimate of the effect of loading weights on the ship.

Inclining experiments

Another facet of rapid adoption of the metacenter was that the first recorded inclining

Table 1: Historical development of stability concepts

No.	Concepts and Methods First Described	Archimedes bef. 212 BC	Stevin 1608	Huygens 1650	Hoste 1697	La Croix 1732-35	Bouguer 1732-46	Euler 1735-49
1.	Conceptual experiments	√						
2.1	Gravity force	√						
2.2	Resulting gravity force of a system of weights	√						
2.3	Buoyancy force, its magnitude and direction	√						
2.4	Hydrostatic equilibrium, Archimedes' law	√	√	√				
3.1	Concept of stable/unstable equilibrium	√		√				
3.2	Test of "small" displacement	√						
4.1	Couple of gravity & buoyancy forces as equilibrium criterion	√		√				
4.2	Calculation of sign of couple for simple shapes	√		√				
4.3	Center of gravity in same vertical as center of buoyancy	√	√	√	√	√	√	√
4.4	Center of gravity may be higher than center of buoyancy				√	√	√	√
4.5	Explicit calculation of ship weight and center of gravity by summing of weights and moments						√	
5.1	Method of wedges for ship stability calculations					√	√	√
5.2	Infinitely small wedges						√	√
5.3	Use of calculus in deriving stability equation						√	√
5.4	Calculation of buoyancy force and direction for arbitrary shapes						√	√
5.4	Use of the metacenter as a stability criterion						√	–
5.5	Naming this point metacenter						√	–
5.6	Stability criterion phrased in metacentric terms						√	–
5.7	Stability criterion based on restoring moment						–	√
6.1	Determination of stability by inclining experiment				√		√	–
7.1	Metacentric curve for finite angles of heel						√	–
7.2	Its founding in the evolute						√	–
7.3	Metacenter calculations for finite angles of heel						√	–

took place in 1748, just two years after BOUGUER published his work. Guillame CLAIRIN-DESLAURIERS, then a junior constructor at Brest, performed the experiment on the newly-built 74-gun ship *Intrépide*, apparently out of curiosity to test the new theory. He hung two 24-pound cannons (each weighing over 2 tonnes) from a buttress built on the side of the ship, and although the buttress broke, he was able to take enough measurements to ascertain that the ship's GM was 1.8m [40].

CHAPMAN and SIMPSON's Rule

Another important contribution to the promulgation of hydrostatic stability calculations came from the Swedish naval constructor and innovative ship designer Fredrik Henrik CHAPMAN. CHAPMAN (1721-1808), the son of the superintendent of the

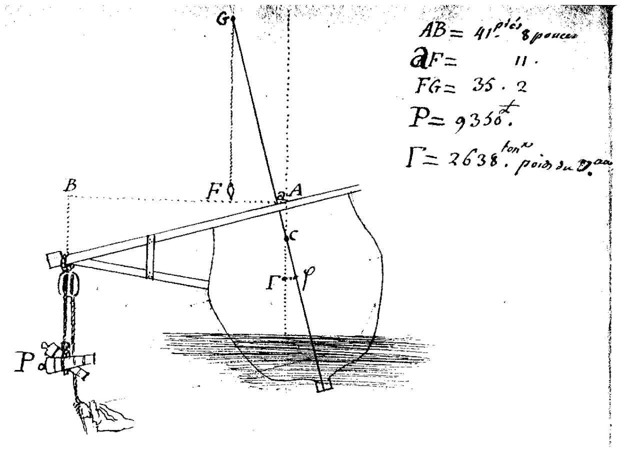

Figure 8: CLAIRIN-DESLAURIERS' drawing of the inclining of *Intrépide* [40].

Göteborg dockyard, was endowed by this background with much practical shipbuilding knowledge. He visited Britain in 1750 to improve his knowledge in mathematics and other fundamentals. He took classes from Thomas SIMPSON (1710-1761), a gifted mathematician trained in NEWTON's tradition and an instructor at the Royal Military Academy in Woolwich near London. SIMPSON reportedly taught CHAPMAN a quadrature rule for planar curves, which yields a numerical approximation of the integral of a continuous function using at least one equally spaced double interval, hence at least three offset points per double interval, or any desired number of further double intervals of the same spacing. This quadrature rule later became known as "SIMPSON's Rule", especially in shipbuilding, although the rule has several historical precursors and SIMPSON never claimed to have invented it. CHAPMAN, after further visits to France and Holland and having picked up much valuable background including the recent findings by BOUGUER on the metacenter, returned in 1757 to Sweden to continue his career as an important ship constructor. In this process he soon made it his habit to calculate ship displacements (1767), also as a function of draft, and assessing the metacenter, applying SIMPSON's Rule for numerical integration. His engineering methods applied in ship design became well known internationally soon after the publication of his book *Tractat om Skepps-Byggeriet* [42] in 1775, where he explained his calculations in much detail (for examples, see Harris [43]). Thus CHAPMAN helped to promulgate the use of the metacenter and to make its calculation a routine matter among naval architects. At the same time SIMPSON's Rule, which offers certain efficiency advantages over the trapezoidal rule for an equal number of intervals, was popularized and is still much favored today.

4.2 Extensions of stability theory

Hydrostatic stability about other axes

In their work on the masting of sailing ships in 1727 it must have occurred to both BOUGUER and EULER that in general the external wind moments would act in an oblique direction to the ship's centerplane so that heeling and trimming would result simultaneously. EULER remembered this open question and in "Scientia Navalis", Part 2, returned to this issue. He derived an expression for the restoring moment about an oblique axis between the axes of heel and trim. The moments of inertia about this axis could be constructed, when the moments of inertia I_T and I_L for transverse and longitudinal stability were known, by using an ellipse of inertia moment construction. Thus this method was able to predict at least the initial hydrostatic response (small angles) to a wind moment in an oblique plane in terms of simultaneous heel and trim angles. EULER mentioned on the side, too, that longitudinal stability was always much greater than transverse stability.

It was some time later that the Spanish naval constructor and engineering scientist Jorge JUAN Y SANTACILIA (1713-1773), who knew BOUGUER and his work very well from the Peruvian expedition and also was in correspondence with several European

Academies, concerned himself not only with ship stability, but also with ship oscillations, especially with pitching motions. In this context he extended BOUGUER's concepts of the metacenter to longitudinal inclinations and first introduced the definition of the longitudinal metacentric radius GM_L. He pointed out that it by far exceeds the transverse metacentric radius. His book "Examen Maritimo" [44], which appeared in 1771, combined the theoretical insights of his days with much practical design experience and became a widely used reference and textbook.

Large angles of heel

A broader extension of stability theory occurred not on the European continent but in Britain. In two papers presented to the Royal Society of London, the British mathematician George ATWOOD examined the inclination of ships at large angles of heel. The first paper [45] of 1796 based on a thorough knowledge of ARCHIMEDES, BOUGUER and EULER, reviews the fundamental physical principles of hydrostatic stability, applied to finite angles of heel. Here ATWOOD investigates the stability properties of homogeneous solids of simple shape (parallelepiped, cylinder, paraboloid) through 360 degrees of rotation as a function of body draft at rest (or specific weight). He finds numerous equilibrium positions (8 or 16), only some of them stable. He develops a shifted wedge volume method for finite heeling angles and reviews the numerical quadrature rules, settling for STIRLING's 3 interval rule. This paper already drew full attention to the fact that stability must be judged over a range of finite, practical heel angles; initial stability alone is inadequate as a stability measure.

In the second paper [46] of 1798, which ATWOOD coauthored with the French constructor Honoré-Sébastien VIAL DU CLAIRBOIS, the investigation was extended to actual ships, and for the first time a numerical analysis of the "righting moments" of ships over a large range of heeling angles was performed. ATWOOD and VIAL DU CLAIRBOIS introduced the term GZ for the "righting arm", which was again numerically evaluated by a wedge volume shift method. This successfully demonstrated the feasibility of numerical stability analysis for actual ships over a range of finite heeling angles. The necessity of performing such calculations rather than just relying on initial stability measures was again underscored. Yet in shipbuilding practice it still took several more decades before the initial difficulties in numerical integration could be overcome by more robust methods and instruments. Even half a century later such stability evaluation had not yet become a routine matter.

It was not before the early 19th century that the knowledge on the metacenter progressed one further step by the work of French constructor and mathematician Charles DUPIN from metacentric curves to metacentric surfaces [47]. DUPIN stated that the ship for finite angles of inclination in whatever direction possesses a surface of centers of buoyancy. This surface at every point has two principal curvatures, thus one can construct two sheets of metacentric surfaces, i.e., evolute surfaces of the buoyancy surface. The two metacenters for the upright condition, M_T and M_L, each is a special point on one of the surfaces.

It is probably fair to say that at this level of perception the issue of hydrostatic stability had been fully resolved in terms of the relationships between ship geometry, physical responses to inclinations, and differential geometry of metacentric surfaces for finite angles of heel.

5. CONCLUSIONS

ARCHIMEDES laid the foundations for the stability of floating systems, introduced a stability measure similar to the righting arm and presented an approach for assessing the ability of a floating inclined solid to right itself.

But his applications were limited to simple geometrical shapes. Fortunately his manuscript “On Floating Bodies” survived in a few copies and became more accessible again by a Latin translation in the 13th c. and also in print after 1500. Yet it took a few more centuries before modern hydrostatic stability was established and could be applied to actual ships, in particular also at the design stage.

The interest in scientific solutions for ship stability gained new momentum for practical and scientific reasons by about 1700:

- The leading navies were getting more concerned about stability risks with increasing ship sizes, gunports that were close to the water and increasingly heavier armament.
- In the beginnings of the Age of Enlightenment, new expectations were raised with regard to the capabilities of science to predict physical phenomena and the performance of technical systems.
- Mathematical breakthroughs occurred in infinitesimal calculus, analytical and numerical methods.
- The modern age of engineering science made rapid progress in mechanics and hydromechanics, including the study of equilibrium and stability of mechanical systems.

All of this contributed to an atmosphere in the scientific community in the early 18th c. that was open to new challenges, also in application to ships. Both BOUGUER and EULER were actively involved in these general developments and certainly exposed to this unique zeitgeist. BOUGUER responded to the recognized problem of hydrostatic stability more as an engineering scientist, EULER reacted rather more like an applied mechanicist and mathematician. Both were able to reformulate and solve this problem in their own unique and original ways.

As our comparisons in this paper have reconfirmed, BOUGUER’s and EULER’s nearly simultaneous work was not only performed quite independently, which was never doubted, but was also distinctly different in approach and justification. Both investigated the ability of the ship to right itself after an infinitesimal heeling displacement. BOUGUER reasoned mainly geometrically and for the inclined ship derived a length measure, the metacentric height, as the decisive geometric measure of initial stability. EULER argued mainly on mechanical grounds and deduced the initial restoring moment as his measure of hydrostatic stability. Despite these contrasting styles of justification, as we appreciate today of course, both stability measures are in fact equivalent and can be converted into each other. We may still benefit from both lines of thought and should be grateful for the insights we owe to these two congenial pioneers.

We have explained why it took several more decades before these landmark discoveries in ship stability were fully accepted and widely applied in shipbuilding practice. But the foundations laid by BOUGER and EULER have remained of lasting value as secure cornerstones in our knowledge for designing safe and stable ships.

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